Low-Frequency Scattering of Dielectric Cylinders

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***Abstract*—A previously developed method of taking the geometric mean of the upper and the lower bounds as the final answer is applied to the problem of low-frequency scattering of dielectric cylinders. Some of the results obtained agree well with those of Mei and Van Bladel and all results are checked numerically to be reasonable for practical applications. The dipole line of linear moment p and p' for the two polarizations of the applied field on the rectangular conducting cylinder is believed to be exact; so are those for the equilateral conducting triangular cylinder and for the conducting regular pentagonal cylinder.**

I. Introduction

IN THIS PAPER we take up the problem of the low-frequency or Rayleigh region scattering of the rectangular cylinder when the wavelength is much greater than the maximum transverse dimensions of the cylinder. Much work has been done on this subject [1]. The results already published do not agree satisfactorily as they should among one another.

In a previous work [2] we have calculated the characteristic impedance of a coaxial line of rectangular outer conductor and circular inner conductor, by finding upper and lower bounds on the characteristic impedance and by taking the mean value of these bounds as the final answer. They are closed form expressions and they are exact. It was discovered empirically that if the geometrical mean is used instead of the arithmetic mean, better results can be obtained. In fact, for the square slab line, the results thus obtained are believed to be the most accurate results yet available [3]. On applying this same method to the low-frequency scattering problem of a rectangular dielectric cylinder, we can take the normalized dipole moment of the inscribed dielectric elliptical cylinder as the low bound to that of the rectangular dielectric cylinder, and take the normalized dipole moment of the circumscribed dielectric cylinder of minimum cross-sectional area as the upper bound to that of the rectangular dielectric cylinder. To check the validity of this method we use the results of another work [4], giving the scattering effect of metallic right polygonal cylinders which are closed form and exact in a quasi-static sense.

II. Previous Works

Our previous work [4] might not have been noticed, so we outline its main points briefly.

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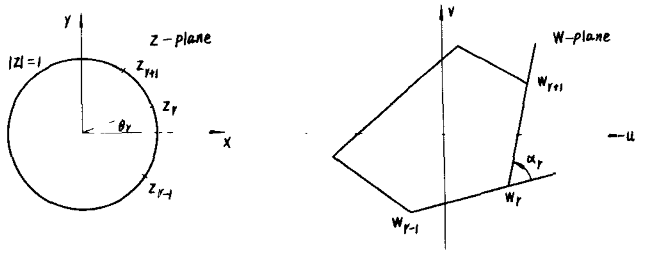

Fig. 1. Transformed circular cylinder and original polygonal cylinder.

A. Scattering by Conducting Rectangular Cylinders

By conformal transformation, we transform the exterior of a polygonal cylinder into the exterior of a unit circular cylinder. In so doing, an incoming uniform field is transformed into a nonuniform field, of which the unit circular cylinder forms a boundary. This uniform field is the first approximation to the incoming plane wave. The potential problem of a circular cylinder in a given external field is then solved.

Let the exterior of a polygon in the u–v plane, as shown in Fig. 1, be transformed into that of a unit circular cylinder, then [3]

$$\frac{dW}{dZ} = C \prod_{r=1}^{N} \left(1 - \frac{Z_r}{Z}\right)^{\alpha_r/\pi} \tag{1}$$

where

$$\sum \alpha_r = 2\pi \qquad \sum \alpha_r Z_r = 0.$$

In this figure, we consider two kinds of polarization: the wave with vector E parallel to the cylinder axis is called "E-wave," while that with vector H parallel to axis is called "H-wave."

Equation (1) transforms the uniform field in the w-plane, $-E_0 u$ or $-E_0 v$, into nonuniform fields, $-E_0 u(r,\theta)$, and $-E_0 v(r,\theta)$, respectively, in the z-plane. In the z-plane the boundary condition at $|z|=1$, i.e.,

$$\varphi|_{z=1} = 0, \qquad H\text{-wave}$$

or

$$\left.\frac{\partial \varphi}{\partial n}\right|_{z=1} = 0, \qquad E\text{-wave}$$

is then made use of to give the required solution.

For a right rectangular cylinder, we have

$$W = C \int_{e^{-i\theta_1}}^{z} \frac{(z-e^{i\theta_1})(z-e^{-i\theta_1})(z-e^{i(\pi-\theta_1)})(z-e^{i(\pi+\theta_1)})^{1/2}}{z^2} \cdot dz + d\theta^{-\theta_0}. \tag{2}$$

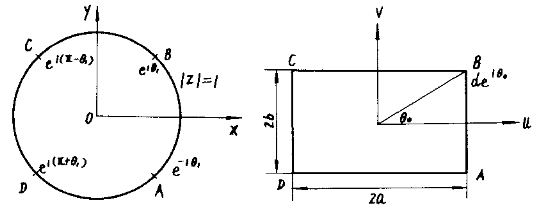

Fig. 2. Transformed circular cylinder and original rectangular cylinder.

Then, from (1), we transform the exterior of the unit circle into that of the rectangular. Here the vertices A, B, C, and D are related in two figures as shown in Fig. 2. The ratio of the long side to the short side being $\tan\theta_0$, we have

$$\frac{b}{a}=\tan\theta_0=\frac{E(\sin\theta_1)-K(\sin\theta_1)\cos^2\theta_1}{E(\cos\theta_1)-K(\cos\theta_1)\sin^2\theta_1} \tag{3}$$

and

$$C=\frac{a}{2\left(E(\cos\theta_1)-K(\cos\theta_1)\sin^2\theta_1\right.}. \tag{4}$$

Now, a uniform field $-E_0u$ along the u-axis in the w-plane will be transformed into a nonuniform field $-u(r,\theta)$ in the z-plane; so in the presence of the conducting cylinder, as the total field we may take ($E_0=1$)

$$\varphi_0=-u(r,\theta)+\sum_n \frac{B_n}{r^n}\cos n\theta.$$

At $r=1$, the boundary condition gives

$$-u(1,\theta)+\sum B_n\cos n\theta=0$$

and B_n is seen to be the cosine Fourier coefficients of $u(1,\theta)$

$$B_1=\frac{2}{\pi}\int_{\theta_1}^{\pi-\theta_1}2C\left(\cos^2\theta_1-\cos^2\theta\right)^{1/2}\sin\theta\,d\theta=2C\cos^2\theta_1$$

$$B_2=\frac{-2}{\pi 2}\int_{\theta_1}^{\pi-\theta_1}-2C\left(\cos^2\theta_1-\cos^2\theta\right)^{1/2}\sin 2\theta\,d\theta=0$$

and so on. In the x–y-plane, the first-order perturbing term is

$$\frac{B_1\cos\theta}{r}=\frac{2C\cos^2\theta_1\cos\theta}{r}$$

but from (1), it is seen that as $z\to\infty$, $W\approx Cz$, so

$$\frac{B_1\cos\theta}{r}\approx\frac{CB_1\cos\psi}{|W|}$$

where ψ is the argument angle in the w-plane. Coming back to the w-plane, we can write the first-order perturbing term as

$$\frac{2C^2\cos^2\theta_1}{W}\cos\psi$$

which is equivalent to the effect of a dipole of moment $2\pi\epsilon_0(2C^2\cos^2\theta_1)$ the direction of the dipole being along the negative u-axis. Here ϵ_0 is the dielectric constant of the medium surrounding the rectangular cylinder.

Putting the value of C from (4) and taking the moment of the dipole term of the perturbing field by the rectangular cylinder to be p, we have finally

$$\frac{p}{\epsilon_0 a^2}=\frac{\pi\cos^2\theta_1}{\left[E(\cos\theta_1)-K(\cos\theta_1)\sin^2\theta_1\right]^2}=\frac{2\pi CB_1}{a^2}. \tag{5}$$

If the external electric field is parallel to the v-axis of Fig. 2, we take

$$\varphi_0=-v(r,\theta)+\sum_n B_n'/r^n\sin n\theta$$

and by the boundary condition $\varphi|_{r=1}=0$, B_n' is seen to be the coefficients of the expansion of the sine Fourier series of $v(1,\theta)$. We find among the others

$$B_1'=\frac{2}{\pi}\int_{-\pi/2}^{\pi/2}v(1,\theta)\sin\theta\,d\theta=\frac{2}{\pi}\int_{-\pi/2}^{\pi/2}\frac{\partial v(1,\theta)}{\partial\theta}\cos\theta s\theta$$

$$=\frac{8C}{\pi}\int_0^{\theta_1}\cos\theta\left(\cos^2\theta-\cos^2\theta_1\right)^{1/2}d\theta=2C\sin^2\theta_1.$$

Again, we take the moment of the dipole term of the perturbing field by the cylinder to be p', then $p'=2\pi\epsilon_0CB_1'=2\pi\epsilon_0C(2C\sin^2\theta_1)$. Putting in (4), we again have

$$\frac{p'}{\epsilon_0a^2}=\frac{\pi\sin^2\theta_1}{\left[E(\cos\theta_1)-K(\cos\theta_1)\sin^2\theta_1\right]^2}=\frac{p'}{\epsilon_0b^2}\frac{b^2}{a^2} \tag{6a}$$

or

$$\frac{p'}{\epsilon_0b^2}=\frac{\pi\sin^2\theta_1}{\left[E(\sin\theta_1)-K(\sin\theta_1)\cos^2\theta_1\right]^2}. \tag{6b}$$

In regard to the problem of incoming E-wave, the problem to be solved is one of a static magnetic field. Let the external magnetic field be along the u-axis; then the resultant magnetic potential may be written as

$$\varphi_0=-u+\sum A_n\frac{\cos n\theta}{r^n}.$$

The boundary condition of the vanishing of the normal component of the magnetic field gives $\partial\varphi_0/\partial r=0$ at $r=1$ in the x–y-plane, so that

$$A_n=\frac{-2}{\pi n}\int_0^{\pi}\frac{\partial u}{\partial r}\cos n\theta\,d\theta=\frac{-2}{\pi n}\int_0^{\pi}\frac{\partial v}{\partial\theta}\cos n\theta\,d\theta.$$

However, when $\theta_1<\theta<-\theta_1$, $v=$constant

$$A_1=\frac{-\psi}{\pi}\int_0^{\theta_1}\frac{\partial v}{\partial\theta}\cos\theta\,d\theta$$

$$=\frac{-8C}{\pi}\int_0^{\theta_1}\left(\cos^2\theta-\cos^2\theta_1\right)^{1/2}\cos\theta\,d\theta=-2C\sin^2\theta_1.$$

Then, if m is taken as the moment of the magnetic dipole effect of the perturbing field, it is given by

$$m=-2\pi\mu_0(2C\sin^2\theta_1)C=-4\pi\mu_0C^2\sin^2\theta_1$$

μ_0 being the permeability of the medium external to the cylinder. Putting in (4), we have

$$\frac{m}{\mu_0a^2}=\frac{-\pi\sin^2\theta_1}{\left[E(\cos\theta_1)-K(\cos\theta_1)\sin^2\theta_1\right]^2}=-\frac{p'}{a^2\epsilon_0}. \tag{7a}$$

When the external magnetic field is parallel to the narrow sides, we take

$$\varphi_0=-v+\sum A'_n\frac{\sin n\theta}{r^n}$$

and then

$$A'_n=\frac{-2}{\pi n}\int_{-\pi/2}^{\pi/2}\frac{\partial v}{\partial r}\sin n\theta\,d\theta$$

where

$$\begin{aligned}A_1&=-\frac{2}{\pi}\int_{-\pi/2}^{\pi/2}\frac{-\partial u}{\partial\theta}\sin\theta\,d\theta\\&=\frac{-8C}{\pi}\int_{\theta_1}^{\pi/2}(\cos^2\theta_1-\cos^2\theta)^{1/2}\sin\theta\,d\theta\\&=-2C\cos^2\theta_1.\end{aligned}$$

The moment of the dipole term is

$$m'=-4\pi\mu_0C^2\cos^2\theta_1 \tag{7b}$$

and by means of (6)

$$\frac{m'}{\mu_0a^2}=\frac{-p}{a^2\epsilon_0}=\frac{-\cos^2\theta_1}{\left[E(\cos\theta_1)-K(\cos\theta_1)\sin^2\theta_1\right]^2}.$$

B. Scattering by Equilateral Conducting Triangular Cylinder

Let the equilateral triangle be shown in the u–v-plane of Fig. 3. From (1) we take

$$W=C\int_1^z\frac{(z^3-1)^{2/3}}{z^2}dz+a$$

to transform the exterior of a unit circle into that of the triangle. From point B we have

$$C=\frac{3}{4}\frac{\sqrt{3}}{3\sqrt{2}}\left(B\frac{5}{6};\frac{5}{6}\right)^{-1}a=0.735a.$$

If incoming is the H-wave, then letting the external field be u-directed, we set

$$\varphi_0=-u+\sum B_n\frac{1}{r^n}\cos n\theta$$

and $\varphi_0=0$ at $r=1$. We find

$$\begin{aligned}B_1&=\frac{2}{\pi}\sqrt[3]{4}\,\frac{\sqrt{3}}{2}C\int_0^{2\pi/3}\left(\sin\frac{3}{2}\theta\right)^{2/3}\sin\theta\,d\theta\\&=\frac{2}{\pi}\sqrt[3]{4}\,\frac{\sqrt{3}}{2}C\int_0^{\sqrt{3}/2}4(3t-4t^3)^{2/3}t\,dt\\&=\frac{9\sqrt{3}}{2}\frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)}{\Gamma(3)}C=0.735a\end{aligned}$$

or

$$\frac{p}{a^2\epsilon_0}=\frac{9\sqrt{3}\,\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)}{2}\frac{9\times3}{16\sqrt[3]{4}}\left(B\left(\frac{5}{6};\frac{5}{6}\right)\right)^{-2}=3.40. \tag{8a}$$

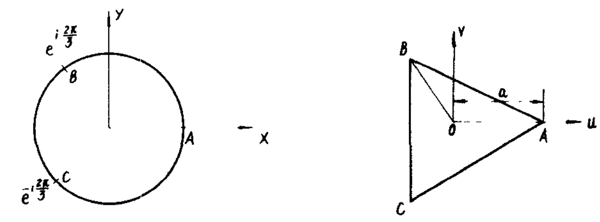

Fig. 3. Transformed circular cylinder and origin equilateral triangular cylinder.

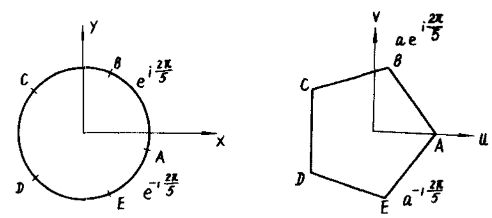

Fig. 4. Transformed circular cylinder and origin regular pentagon cylinder.

If incoming is the E-wave directed along the u-axis, so that $\partial\varphi/\partial r|_{r=1}=0$, we have as before

$$\varphi_0=-u+\sum\frac{A_n}{r^n}\cos n\theta$$

$$A_1=0.775a.$$

Therefore we find the equivalent magnetic dipole with moment

$$\frac{m}{a^2}=\frac{2\pi A_1C}{a^2}=3.58.$$

C. Scattering by Regular Conducting Pentagonal Cylinders

From (1) let us take

$$W=C\int_1^z\frac{(z^5-1)^{2/5}}{z^2}dz+a$$

to transform the exterior of the unit circle into that of the regular pentagon (see Fig. 4)

$$C=2\sin\frac{\pi}{5}\cdot a\left(\frac{2}{5},2^{4/5}\cdot B\left(\frac{7}{10};\frac{7}{10}\right)\right)^{-1}=0.89a.$$

When the external electric field is along the u-axis, then

$$B_1=\frac{2}{\pi}\int_0^{\pi}u\cos\theta\,d\theta=\frac{-2}{\pi}\int_0^{\pi}\sin\theta\frac{du}{d\theta}d\theta.$$

For $0<\theta<2\pi/5$

$$\begin{aligned}\frac{\partial u}{\partial\theta}&=\mathrm{Re}\,\frac{dW}{d\theta}=\mathrm{Re}\left\{Ce^{i7\pi/10}2^{2/5}\left(\sin\frac{5}{2}\theta\right)^{2/5}\right\}\\&=-\sin\frac{\pi}{5}\cdot C\cdot2^{2/5}\left(\sin\frac{5}{2}\theta\right)^{2/5}.\end{aligned}$$

But for $2\pi/5<\theta<4\pi/5$

$$\frac{du}{d\theta}=-C2^{2/5}\cos\frac{\pi}{10}\left(-\sin\frac{5}{2}\theta\right)^{2/5}$$

and for $4\pi/5<\theta<\pi$

$$du/d\theta=0$$

so we have

$$B_1=\frac{2}{\pi}\left[\int_0^{2\pi/5}2^{2/5}C\sin\frac{\pi}{5}\left(\sin\frac{5}{2}\theta\right)^{2/5}\sin\theta\,d\theta\right.$$
$$\left.+\int_{2\pi/5}^{4\pi/5}C2^{2/5}\cos\frac{\pi}{10}\left(-\sin\frac{5}{2}\theta\right)^{2/5}\sin\theta\,d\theta\right]$$
$$=\frac{2\cdot 2^{2/5}\cdot C}{\pi}\left[\frac{3}{2}\sin\frac{\pi}{5}\int_0^{2\pi/5}\left(\sin\frac{5}{2}\theta\right)^{2/5}\sin\theta\,d\theta\right.$$
$$\left.+\cos^2\frac{\pi}{10}\int_0^{2\pi/5}\left(\sin\frac{5}{2}\theta\right)^{2/5}\cos\theta\,d\theta\right].$$

If the external u-directed field is a magnetic field, we have

$$\varphi_0=-u+\sum A_n\frac{\cos n\theta}{r^n}$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=1}-\left.\sum A_n\frac{n\cos n\theta}{r^{n+1}}\right|_{r=1}=0$$

and hence

$$A_n=\frac{-2}{n\pi}\int_0^{\pi}\frac{\partial u}{\partial r}\cos n\theta\,d\theta=\frac{-2}{n\pi}\int_0^{\pi}\frac{\partial v}{\partial\theta}\cos n\theta\,d\theta.$$

For $0<\theta<2\pi/5$

$$\frac{\partial v}{\partial\theta}=2^{2/5}\cdot C\cos\frac{\pi}{5}\left(\sin\frac{5}{2}\theta\right)^{2/5}$$

and for $2\pi/5<\theta<4\pi/5$

$$\frac{\partial v}{\partial\theta}=-C2^{2/5}\sin\frac{\pi}{10}\left(-\sin\frac{5\theta}{2}\right)^{2/5}$$

and finally, for $4\pi/5<\theta<\pi$

$$\frac{\partial v}{\partial\theta}=-C2^{2/5}\left(\sin\frac{5\theta}{2}\right)^{2/5}.$$

So we have

$$A_1=\frac{-2}{\pi}\left[\int_0^{2\pi/5}\cos\frac{\pi}{5}\left(\sin\frac{5\theta}{2}\right)^{2/5}\cos\theta\,d\theta\right.$$
$$-\int_{2\pi/5}^{4\pi/5}\sin\frac{\pi}{10}\left(-\sin\frac{5}{2}\theta\right)^{2/5}\cos\theta\,d\theta$$
$$\left.-\int_{4\pi/5}^{\pi}\left(\sin\frac{5\theta}{2}\right)^{2/5}\cos\theta\,d\theta\right]\cdot 2^{2/5}C$$

$$A_1=\frac{-2\cdot 2^{2/5}\cdot C}{\pi}\left[\sin\frac{\pi}{5}\int_0^{2\pi/5}\left(\sin\frac{5\theta}{2}\right)^{2/5}\sin\theta\,d\theta\right.$$
$$\left.+\left(\frac{3}{2}\cos\frac{\pi}{5}-\sin^2\frac{\pi}{10}\right)\int_0^{2\pi/5}\left(\sin\frac{5\theta}{2}\right)^{2/5}\cos\theta\,d\theta\right].$$

We have to evaluate two integrals:

$$I_1=\int_0^{2\pi/5}\left(\sin\frac{5\theta}{2}\right)^{2/5}\sin\theta\,d\theta$$
$$=4\int_0^{\sin\pi/5}(5t-20t^3+16t^5)^{2/5}t\,dt$$
$$=4\cdot 2^{3/5}\int_0^{\sin 2\pi/5}u^{1/5}\left(u^2-\frac{5}{4}u+\frac{5}{16}\right)^{2/5}du.$$

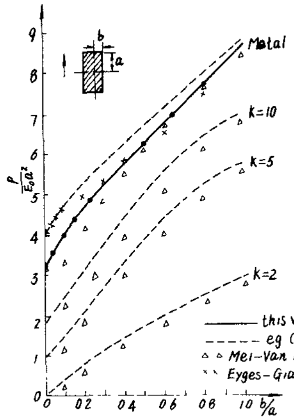

Fig. 5. Normalized linear dipole moment of a dielectric rectangular cylinder, applied field parallel to the long side.

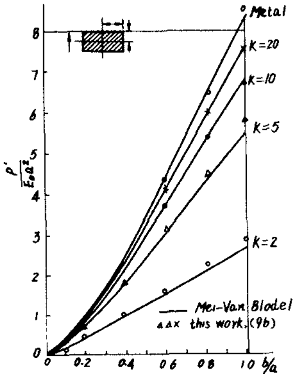

Fig. 6. Normalized linear dipole moment of a dielectric rectangular cylinder, applied field parallel to the short side.

However

$$\sin^2\frac{\pi}{5}=\frac{5-\sqrt{5}}{8};\ \left(u^2-\frac{5}{4}u+\frac{5}{16}\right)$$
$$=\left(\frac{5+\sqrt{5}}{8}-u\right)\left(\frac{5-\sqrt{5}}{8}-u\right).$$

So

$$I_1=4\cdot 2^{3/5}\int_0^{5-5/8}u^{1/5}\left(\frac{5+\sqrt{5}}{8}-u\right)^{2/5}$$
$$\cdot\left(\frac{5-\sqrt{5}}{8}-u\right)^{2/5}du$$
$$=\frac{2^{3/5}}{16}(5-\sqrt{5})(5-\sqrt{5})^{3/5}(5+\sqrt{5})^{2/5}$$
$$\cdot B\left(\frac{6}{5};\frac{7}{5}\right)F\left(-\frac{2}{5};\frac{6}{5};\frac{13}{5};\frac{5-\sqrt{5}}{5+\sqrt{5}}\right)=0.63$$

and

$$I_2=\int_0^{2\pi/5}\left(\sin\frac{5}{2}\theta\right)^{2/5}\cos\theta\,d\theta$$
$$=\int_0^{2\pi/5}\left(\sin\frac{5\theta}{2}\right)^{2/5}\left(1-2\sin^2\frac{\theta}{2}\right)d\theta$$

where the first term has been evaluated

$$\int_0^{2\pi/5}\left(\sin\frac{5\theta}{2}\right)^{2/5} d\theta\approx 1.00.$$

Thus we have

$$I_3=\int_0^{2\pi/5}\left(\sin\frac{5\theta}{2}\right)^{2/5}\sin^2\frac{\theta}{2}d\theta$$

$$=2\int_0^{\sin\pi/5}\frac{(5t-20t^3+(6t^5)^{2/5}t^2\,dt}{\sqrt{1-t^2}}$$

$$=\int_0^{\sin^2\pi/5}\frac{t^{7/10}(t-20t+16t^2)^{2/5}}{(1-t)^{1/2}}dt$$

$$=2\cdot 2^{3/5}\int_0^{5-\sqrt{5}/8}t^{7/10}\left(\frac{5+\sqrt{5}}{8}-t\right)^{2/5}$$

$$\cdot\left(\frac{5-\sqrt{5}}{8}-t\right)^{2/5}(1-t)^{-1/2}dt$$

which, for $5+\sqrt{5}/8=0.905\approx 1$, goes nearly into

$$I_3=2\cdot 2^{3/5}\int^{5-\sqrt{5}/8}t^{7/10}\left(\frac{5-\sqrt{5}}{8}-t\right)^{2/5}(1-t)^{-1/10}dt$$

$$=2\cdot 2^{3/5}\left(\frac{5-\sqrt{5}}{8}\right)^{7/10+1+2/5}\int_0^1 u^{7/10}(1-u)^{2/5}$$

$$\cdot\left(1-\frac{5-\sqrt{5}}{8}\right)^{-1/10}du$$

$$=2\cdot 2^{3/5}\left(\frac{5-\sqrt{5}}{8}\right)^{2+1/10}B\left(\frac{17}{10};\frac{7}{5}\right)$$

$$\cdot F\left(\frac{1}{10};\frac{17}{10};\frac{31}{10};\frac{5-\sqrt{5}}{8}\right)\approx 0.123$$

and so

$$I_2=1-2\times 0.123=0.754.$$

From this, we find A_1 and B_1

$$B_1=\frac{2}{\pi}2^{3/5}C(1.5\sin 36°\times 0.63+\cos^2 18°\times 0.754)=0.92a.$$

Therefore

$$\frac{p}{a^2\epsilon_0}=\frac{2\pi B_1C}{a^2}=5.15 \qquad (8b)$$

and

$$A_1=\frac{-2\cdot 2^{2/5}C}{\pi}$$

$$\cdot\left[\sin 36°\times 0.63+(1.5\cos 36°-\sin^2 18°)\times 0.754\right]$$

$$=-0.908a.$$

Hence

$$\frac{m}{\mu_0a^2}=\frac{2\pi A_1C}{a^2}=5.10.$$

We treat only the case of the u-directed electric or magnetic field for the cases of regular triangular and pentagonal cylinders, for a field of arbitrary orientation can be resolved into two components parallel; respectively, to ***OA*** and ***OB***. Owing to symmetry, the scattering effect on the component parallel to ***OB*** is the same as that on the component parallel to ***OA***. Scattering by other polygonal cylinders can be treated in the same way. However, the evaluation of coefficients is much more tedious.

III. The Scattering of Rectangular Dielectric Cylinders

Now we take up the problem of the scattering at low frequencies of the rectangular dielectric cylinder by taking as the final answer to this problem the geometric mean of the effect of inscribed elliptic dielectric cylinder of a dipole line of linear moment p_i and the effect of the circumscribed elliptic dielectric cylinder of minimum cross-sectional area of a dipole line of linear moment p_c. For a rectangular of sides $2a$ and $2b$, it can easily be proven that the circumscribed ellipse of minimum cross-sectional area is the one with major and minor axes of $\sqrt{2}\,2a$ and $\sqrt{2}\,2b$, i.e.,

$$\frac{x^2}{2a^2}+\frac{y^2}{2b^2}=1.$$

While the values of p_i and p_c (obviously $p_c=2p_i$) can be obtained from the existing literature [5]. Thus, an elliptic dielectric cylinder in the w-plane of Fig. (2) in a uniform field parallel to its major axis will produce a perturbing potential of

$$\Phi_1=-E_0\left\{w-\sqrt{w^2-(a^2-b^2)}\right\}\frac{\frac{b}{a}(k-1)}{\left(1-\frac{b}{a}\right)\left(1+k\frac{b}{a}\right)}$$

$$=-E_0\frac{1}{2}a^2\frac{\frac{b}{a}(k-1)\left(1+\frac{b}{a}\right)}{\left(1+\frac{b}{a}k\right)w}+\cdots$$

which is equivalent to the effect of a dipole moment of

$$p_i=\pi(a^2\epsilon_0)\frac{\frac{b}{a}(k-1)}{1+\frac{b}{a}k}\left(1+\frac{b}{a}\right)$$

E_0 being taken as unity. If the external uniform field is along the minor axis of the elliptic dielectric cylinder, the perturbing potential is given by

$$\Phi_2=-jE_0\left\{w-\sqrt{w^2-(a^2-b^2)}\right\}\frac{\frac{b}{a}(k-1)}{\left(1-\frac{b}{a}\right)\left(k+\frac{b}{a}\right)}$$

$$=-jE_0\frac{1}{2}a^2\frac{\frac{b}{a}(k-1)\left(1+\frac{b}{a}\right)}{\left(k+\frac{b}{a}\right)w}$$

which is equivalent to the effect of a dipole moment of

$$p_i' = \pi(a^2\epsilon_0)\frac{\frac{b}{a}(k-1)}{k+b/a}\left(1+\frac{b}{a}\right), \qquad E_0 = 1.$$

Since the dipole moment of the circumscribed elliptical dielectric cylinder is twice that of the inscribed one, so the final answer, being taken as their geometric mean is as follows.

E_0 parallel to major axis

$$\frac{p_i}{\epsilon a^2} = \sqrt{2}\,\pi\frac{\frac{b}{a}(k-1)}{1+b/ak}\left(1+\frac{b}{a}\right) \tag{9a}$$

E_0 parallel to minor axis

$$\frac{p_i'}{\epsilon a^2} = \sqrt{2}\,\pi\frac{\frac{b}{a}(k-1)}{k+b/a}\left(1+\frac{b}{a}\right). \tag{9b}$$

The dielectric cylinder becomes the conducting cylinder when $k \to \infty$ so we plot (9a) and (9b), in Figs. 5 and 6 starting from $k \to \infty$ down to $k=2$, shown are some results from Eyges and Gianino [1] and from Mei and Van Bladel [6]. In the cases of external field parallel to the minor axis, our results agree very well with those previous results, while in the other case, the agreement is only fair. In fact, for $b/a=1$, $\cos\theta_1 = 1/\sqrt{2}$, (5a) gives a result of

$$\frac{p}{\epsilon_0 a^2} = \frac{\pi/2}{\left\{\frac{\pi}{4}/k\left(\frac{1}{2}\right)\right\}^2} = 8.754$$

while the result read from [6] is only about 8.4. For $k=\infty$, (9a) and (9b) both give a value of $\sqrt{2}\,(2\pi) = 8.8858$, 1.33 percent higher than the exact value of 8.754, while that from [6] is 5.555 percent lower so we might say the results (9a) and (9b) are of practical value.

IV. Scattering of Other Equilateral Polygonal Cylinders

The scattering problems of Figs. 3 and 4 and this kind may be grouped together by finding the geometric mean of the radii of the inscribed and the circumscribed circles considering that the scattering effect of the equilateral polygonal cylinder is identical to an equivalent cylinder of radius equal to this mean value, so the final dipole moment is

$$\frac{p}{\epsilon_0 a^2} = \frac{k-1}{k+1}2\pi\left(\cos\frac{\pi}{n}\right) \tag{10}$$

where n is the number of sides of the equilateral polygon, a is the distance between its center and one of its vertics. To see the validity of (10), take the case of Figs. 3 and 4, first let $k \to \infty$ thus, for the former, $n=3$

$$\frac{p}{\epsilon_0 a^2} = \pi = 3.14 \tag{11}$$

while the exact value is 3.40, so (11) is 7.65 percent too low, and for $n=5$, (10) gives

$$\frac{p}{\epsilon_0 a^2} = (2\pi)\cos 36° = 5.0832 \tag{12}$$

while the exact value is 5.15 of Fig. 4 so (12) is 1.29 percent lower than the exact value. For $n=4$, (10) gives

$$\frac{p}{\epsilon_0\left(\frac{a}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}}(2\pi). \tag{13}$$

Here a is the half-side, and (13) is 1.325 percent lower than the exact value. It is seen that (10) is closer to the exact value the higher is the number of sides n.

V. Conclusion

We have applied the method of taking the geometric mean of the upper and lower bounds of dipole moment of the dielectric cylinder as the final answer to the scattering effect of this dielectric cylinder. This method has been proved to be satisfactory in the calculation of the characteristic impedances of transmission lines of special cross sections. By the numerical checks we have done in this paper, we believe that this method is equally useful in problems of low frequency scattering.

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